THE ONE-DIMENSIONAL SCHRÖDINGER OPERATORS WITH SINGULAR PERIODIC POTENTIALS

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Abstract. The one-dimensional Schrödinger operators

$$S(q)u := -u'' + q(x)u, \quad u \in \mathrm{Dom}\,(S(q))$$

with real-valued 1-periodic singular potentials $q(x) \in H_{per}^{-1}(\mathbb{R})$ are studied on the Hilbert space $L_2(\mathbb{R})$. An equivalence of five basic definitions for the operators $S(q)$ and their self-adjointness are established. A new proof of spectral continuity of the operators $S(q)$ is found. Endpoints of spectral gaps are precisely described.

1. Introduction

On the complex Hilbert space $L_2(\mathbb{R})$ we consider the one-dimensional Schrödinger operators

$$S(q)u := -u'' + q(x)u, \quad u \in \mathrm{Dom}\ (S(q)) \tag{1}$$

with real-valued 1-periodic distributional potentials $q(x)$, so called the Hill-Schrödinger operators.

Under the assumption

$$q(x) = \sum_{k\in 2\mathbb{Z}} \widehat{q}(k)e^{ik\pi x} \in H_{per}^{-1}(\mathbb{R},\mathbb{R}), \tag{2}$$

i.e., when

$$\sum_{k\in 2\mathbb{Z}} (1+|k|)^{-2}|\widehat{q}(k)|^2 < \infty, \quad \text{and} \quad \widehat{q}(k) = \overline{\widehat{q}(-k)} \quad \forall k \in 2\mathbb{Z},$$

the Hill-Schrödinger operators $S(q)$ can be well defined on the Hilbert space $L_2(\mathbb{R})$ in the following different ways:

- as minimal/maximal quasi-differential operators $S_{min}(q)/S_{max}(q)$;
- as Friedrichs extensions $S_F(q)$ of quasi-differential operators $S_{min}(q)$;
- as form-sum operators $S_{form}(q)$;
- as a sequence limits $S_{lim}(q)$ of the Hill-Schrödinger operators with smooth periodic potentials in the norm resolvent sense.

Hryniv and Mykytyuk [HrMk], Djakov and Mityagin [DjMt] studied Friedrichs extensions $S_F(q)$, and Korotyaev [Krt] treated form-sum operators $S_{form}(q)$. We propose join together these results showing an equivalence of all definitions.

More precisely, we will prove the following statements.

Theorem A (Theorem 14). *The Hill-Schrödinger quasi-differential operators $S_{max}(q)$ with distributional potentials $q(x) \in H_{per}^{-1}(\mathbb{R},\mathbb{R})$ are self-adjoint.*

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Theorem B (Corollary 15, Corollary 16, Theorem 18)**.** *Quasi-differential operators $S_{min}(q)$ and $S_{max}(q)$, Friedrichs extensions $S_F(q)$, form-sum operators $S_{form}(q)$ and operators $S_{lim}(q)$ coincide.*

In the paper [HrMk, Theorem 3.5] the authors tried to show that the operators $S_{max}(q)$ and $S_F(q)$ coincide. But proof of this assertion is incorrect. Our proofs of Theorem A and Theorem B base on a different idea (see Lemma 5).

The equality $S(q) = S_{lim}(q)$ together with the classical Birkhoff-Lyapunov Theorem allow to prove the following statement.

Theorem C (Theorem 19)**.** *The Hill-Schrödinger operators $S(q)$ with distributional potentials $q(x) \in H_{per}^{-1}(\mathbb{R}, \mathbb{R})$ have continuous spectra with a band and gap structure such that the endpoints $\{\lambda_0(q), \lambda_k^{\pm}(q)\}_{k=1}^{\infty}$ of spectral gaps satisfy the inequalities:*

$$-\infty < \lambda_0(q) < \lambda_1^-(q) \leq \lambda_1^+(q) < \lambda_2^-(q) \leq \lambda_2^+(q) < \dots .$$

Moreover, the endpoints of spectral gaps for even/odd numbers $k \in \mathbb{Z}_+$ are periodic/semiperiodic eigenvalues of the problems on the interval $[0, 1]$,

$$S_{\pm}(q)u = -u'' + q(x)u = \lambda u, \quad u \in \operatorname{Dom}(S_{\pm}(q)).$$

It is interested to remark that the last assertion is nontrivial and for the more singular δ'-interactions,

$$q(x) = \sum_{k\in\mathbb{Z}} \beta\, \delta'(x-k) \notin H_{per}^{-1}(\mathbb{R}), \quad \beta < 0,$$

an unusual situation, when the endpoints of spectral gaps for even/odd numbers $k \in \mathbb{Z}_+$ are *semiperiodic/periodic* eigenvalues of the problems on the interval $[0, 1]$, is possible [Alb, Theorem III.3.6].

In the closely related paper [HrMk] Hryniv and Mykytyuk established that spectra of the operators $S(q)$ are absolutely continuous.

2. Preliminaries

2.1. **Sobolev spaces.** Let us denote by $\mathfrak{D}'_1(\mathbb{R})$ the Schwartz space of 1-periodic distributions defined on the whole real axis $\mathbb{R}$ (see, for an example, [Vld]). For a detail characteristic of 1-periodic distributions we introduce Sobolev spaces.

So, Sobolev spaces $H_{per}^s(\mathbb{R})$, $s \in \mathbb{R}$ of 1-periodic functions/distributions are defined by means of their Fourier coefficients:

$$H_{per}^s(\mathbb{R}) := \left\{ f = \sum_{k\in 2\mathbb{Z}} \widehat{f}(k) e^{ik\pi x} \,\middle|\, \| f \|_{H_{per}^s(\mathbb{R})} < \infty \right\},$$

$$\| f \|_{H_{per}^s(\mathbb{R})} := \left(\sum_{k\in 2\mathbb{Z}} \langle k \rangle^{2s} |\widehat{f}(k)|^2 \right)^{1/2}, \quad \langle k \rangle := 1 + |k|,$$

$$\widehat{f}(k) := \langle f, e^{ik\pi x} \rangle_{L_{2,per}(\mathbb{R})}, \quad k \in 2\mathbb{Z},$$

$$2\mathbb{Z} := \{ k \in \mathbb{Z} \mid k \equiv 0 \ (\operatorname{mod} 2) \}.$$

Sesqui-linear form $\langle \cdot, \cdot \rangle_{L_{2,per}(\mathbb{R})}$ pairs the dual, respectively $L_{2,per}(\mathbb{R})$, spaces $H_{per}^s(\mathbb{R})$ and $H_{per}^{-s}(\mathbb{R})$, and it is an extension by continuity the $L_{2,per}(\mathbb{R})$-inner product [Brz, GrGr],

$$\langle f, g \rangle_{L_{2,per}(\mathbb{R})} := \int_0^1 f(x)\overline{g(x)}\, dx \quad \forall f, g \in L_{2,per}(\mathbb{R}).$$

It should be noted that

$$H_{per}^0(\mathbb{R}) = L_{2,per}(\mathbb{R}),$$

and that by $\mathfrak{D}'_1(\mathbb{R},\mathbb{R})$ and $H^s_{per}(\mathbb{R},\mathbb{R})$, $s\in\mathbb{R}$ are denoted *real-valued* 1-periodic distributions from the correspondent spaces,

$$\mathfrak{D}'_1(\mathbb{R},\mathbb{R}) := \{ f(x)\in\mathfrak{D}'_1(\mathbb{R}) \,|\, \mathrm{Im} f(x) = 0 \},$$
$$H^s_{per}(\mathbb{R},\mathbb{R}) := \left\{ f(x)\in H^s_{per}(\mathbb{R}) \,|\, \mathrm{Im} f(x) = 0 \right\}.$$

Note that $\mathrm{Im} f(x)=0$ for a 1-periodic distribution $f(x)\in\mathfrak{D}'_1(\mathbb{R})$ means that

$$\widehat{f}(2k) = \overline{\widehat{f}(-2k)} \quad \forall k\in\mathbb{Z}.$$

2.2. **Quasi-differential equations.** The differential expressions in the right-hand of the (1) by introducing quasi-derivatives:

$$u^{[1]}(x) := u'(x) - Q(x)u(x), \qquad \langle q,\varphi\rangle = -\langle Q,\varphi'\rangle \quad \forall\varphi\in C^\infty_{comp}(\mathbb{R}),$$

may be re-written as quasi-differential ones [SvSh1, SvSh2],

$$l_Q[u] := -(u'-Qu)' - Q(u'-Qu) - Q^2u,$$

which are well defined if $u, u^{[1]}\in W^1_{1,loc}(\mathbb{R})$ [Nai].

Proposition 1 (Existence and Uniqueness Theorem). *Let $\lambda\in\mathbb{C}$, and $f(x)\in L_{1,loc}(\mathbb{R})$. Then for any complex numbers $c_0,\ c_1\in\mathbb{C}$ and arbitrary $x_0\in\mathbb{R}$ the quasi-differential equation*

$$l_Q[u] = \lambda u + f, \quad \lambda\in\mathbb{C},\ f\in L_{1,loc}(\mathbb{R}) \tag{3}$$

has one and only one solution $u\in W^1_{1,loc}(\mathbb{R})$ with the initial conditions

$$u(x)\,|_{x=x_0} = c_0, \quad u^{[1]}(x)\,|_{x=x_0} = c_1.$$

With the quasi-differential equation (3) it is related the normal 2-dimensional system of the first order differential equations with the locally integrable coefficients,

$$\begin{pmatrix} u_1\\ u_2\end{pmatrix}' = \begin{pmatrix} Q & 1\\ -\lambda-Q^2 & -Q\end{pmatrix}\begin{pmatrix} u_1\\ u_2\end{pmatrix} + \begin{pmatrix} 0\\ -f\end{pmatrix},$$

where $u_1(x) := u(x)$, $u_2(x) := u^{[1]}(x)$.

Then Proposition 1 follows from [Nai, Theorem 1, §16], also see [AhGl].

Lemma 2 (Lagrange Formula). *Let $u(x)$ and $v(x)$ be functions such that quasi-differential expressions $l_Q[\cdot]$ are well defined. Then the Lagrange formula*

$$l_Q[u]\overline{v} - u l_Q[\overline{v}] = \frac{d}{dx}[u,v]_x$$

holds, where the sesqui-linear forms $[u,v]_x$ are defined in the following fashion:

$$[u,v]_x := u(x)\overline{(v'(x)-Q(x)v(x))} - (u'(x)-Q(x)u(x))\,\overline{v(x)}.$$

Proof. Under the assumption $u(x)$ and $v(x)$ are such that

$$u, u'-Qu\in W^1_{1,loc}(\mathbb{R}) \quad\text{and}\quad v, v'-Qv\in W^1_{1,loc}(\mathbb{R}).$$

Then we have

$$\begin{aligned}
\frac{d}{dx}[u,v]_x &\equiv \frac{d}{dx}\left(u\overline{(v'-Qv)} - (u'-Qu)\,\overline{v}\right)\\
&= u'\overline{(v'-Qv)} + u\overline{(v'-Qv)'} - (u'-Qu)'\,\overline{v} - (u'-Qu)\,\overline{v'}\\
&= l_Q[u]\overline{v} - u l_Q[\overline{v}] + Qu'\overline{v} - Qu\overline{v'} + u'\overline{(v'-Qv)} - (u'-Qu)\,\overline{v'}\\
&= l_Q[u]\overline{v} - u l_Q[\overline{v}]
\end{aligned}$$

taking into account that under the made assumptions

$$u'\overline{v'},\ Q^2u\overline{v},\ Qu'\overline{v},\ Qu\overline{v'} \in L_{1,loc}(\mathbb{R}).$$

The proof is complete. □

Integrating both parts of the Lagrange Formula over the compact interval $[\alpha,\beta] \Subset \mathbb{R}$ we obtain the Lagrange Identity in an integral form,

$$\int_\alpha^\beta l_Q[u]\overline{v}\,dx - \int_\alpha^\beta u l_Q[\overline{v}]\,dx = [u,v]_\alpha^\beta, \tag{4}$$

where

$$[u,v]_\alpha^\beta := [u,v]_\beta - [u,v]_\alpha.$$

2.3. **Quasi-differential operators on a finite interval.** Here, we due to Savchuk and Shkalikov [SvSh1] give a short review of results related with Sturm-Liouville operators with distributional potentials defined on a finite interval.

On the Hilbert space $L_2(0,1)$ we consider the Sturm-Liouville operators

$$L(q)u := -u'' + q(x)u, \quad u \in \mathrm{Dom}\ (L(q))$$

with real-valued distributional potentials $q(x) \in H^{-1}([0,1],\mathbb{R})$, i.e., when

$$Q(x) = \int q(\xi)\,d\xi \in L_2((0,1),\mathbb{R}).$$

Set

$$\begin{aligned} L_{max}(q)u &:= l_Q[u], \\ \mathrm{Dom}(L_{max}(q)) &:= \left\{u \in L_2(0,1) \mid u, u' - Qu \in W_1^1[0,1],\ l_Q[u] \in L_2(0,1)\right\}, \end{aligned}$$

and

$$\begin{aligned} \dot{L}_{min}(q)u &:= l_Q[u], \\ \mathrm{Dom}(\dot{L}_{min}(q)) &:= \{u \in \mathrm{Dom}(L_{max}(q)) \mid \operatorname{supp} u \Subset [0,1]\}. \end{aligned}$$

We also consider the operators

$$\begin{aligned} L_{min}(q)u &:= l_Q[u], \\ \mathrm{Dom}(L_{min}(q)) &:= \left\{u \in \mathrm{Dom}(L_{max}(q)) \mid u^{[j]}(0) = u^{[j]}(1) = 0,\ j = 0,1\right\}. \end{aligned}$$

Proposition 3 ([SvSh1])**.** *Let suppose that $q(x) \in H^{-1}([0,1],\mathbb{R})$. Then the following statements are fulfilled:*

(I) *Operators $L_{min}(q)$ are densely defined on the Hilbert space $L_2(0,1)$.*

(II) *Operators $L_{min}(q)$ and $L_{max}(q)$ are mutually adjoint,*

$$L_{min}^*(q) = L_{max}(q), \qquad L_{max}^*(q) = L_{min}(q).$$

In particular, operators $L_{min}(q)$ and $L_{max}(q)$ are closed.

In Statement 4, which proof is given in Appendix A.1, we establish relationships between operators $\dot{L}_{min}(q)$ and $L_{min}(q)$.

Statement 4. *Operators $L_{min}(q)$ are closures of operators $\dot{L}_{min}(q)$,*

$$L_{min}(q) = (\dot{L}_{min}(q))^{\sim} = \dot{L}_{min}^{**}(q).$$

3. Main results

3.1. Principal lemma. The following operator-theory statement is an essential point of our approach. It has two important applications in this section.

Lemma 5. *Let A be a densely defined and closed on a complex Banach space X linear operator, and let B be a bounded on X linear operator, such that*

(a) $BA \subset AB$ *(A and B commute);*
(b) $\sigma_p(B) = \varnothing$ *(point spectrum $\sigma_p(B)$ of operator B is empty).*

Then the operator A has no eigenvalues of a finite multiplicity.

Proof. Let suppose that the operator A has eigenvalue $\lambda \in \sigma_p(A)$ of a finite multiplicity, and let G_λ be a correspondent eigenspace.

Further, let f be an eigenvector of the operator A,

$$Af = \lambda f, \qquad f \in G_\lambda,$$

Then we have

$$A(Bf) = B(Af) = \lambda(Bf), \qquad f \in G_\lambda,$$

from where one may conclude that

$$BG_\lambda \subset G_\lambda.$$

Taking into account that under the assumption $\dim(G_\lambda) \in \mathbb{N}$ from the latter we obtain that point spectrum $\sigma_p(B)$ of the operator B is not empty. This contradicts to the condition (b).

The proof is complete. □

Remark 6. The condition (b) is valid if the space $X = L_p(\mathbb{R}, \mathbb{C})$, $1 \le p < \infty$, and B be a shift operator,

$$B : \, y(x) \mapsto y(x+T), \, T > 0.$$

Indeed, the operator B is an unitary one on the space $X = L_p(\mathbb{R}, \mathbb{C})$. Therefore

$$\sigma_p(B) \subset \sigma(B) = \{\lambda \in \mathbb{C} \mid |\lambda| = 1\},$$

and the equality

$$By(x) = \lambda y(x) = y(x+T), \quad y(x) \not\equiv 0, \quad |\lambda| = 1$$

implies that the function $|y(x)|$ is T-periodic. Then $y(x) \notin L_p(\mathbb{R}, \mathbb{C})$, and one may conclude that $\sigma_p(B) = \varnothing$.

The condition (a) means in this case that the operator A is T-periodic on the line.

3.2. Self-adjointness of the Hill-Schrödinger operators with distributional potentials. Under the assumption (2) the distributional potentials $q(x)$ have got the representations

$$q(x) = C + Q'(x)$$

with

$$C = \widehat{q}(0)$$

and

$$Q(x) = \sum_{k \in 2\mathbb{Z} \setminus \{0\}} \frac{1}{ik\pi} \widehat{q}(2k) e^{ik\pi x} \in L_{2,per}(\mathbb{R}, \mathbb{R})$$

such that

$$\langle q, \varphi \rangle = -\langle Q, \varphi' \rangle \quad \forall \varphi \in C^\infty_{comp}(\mathbb{R}),$$

see [DjMt, Proposition 1], [Vld]. Here, by $\langle f, \cdot \rangle$, $f \in \mathfrak{D}'(\mathbb{R})$ we denote sesqui-linear functionals over the space $C^\infty_{comp}(\mathbb{R})$.

Remark 7. Without loss of any generality throughout of the remainder of the paper we will assume that

$$\widehat{q}(0) = 0.$$

Then the Hill-Schrödinger operators can be well defined on the Hilbert space $L_2(\mathbb{R})$ as quasi-differential ones [SvSh1, SvSh2] by means of quasi-expressions,

$$l_Q[u] = -(u' - Qu)' - Q(u' - Qu) - Q^2 u.$$

Set

$$\begin{aligned} S_{max}(q)u &:= l_Q[u], \\ \mathrm{Dom}(S_{max}(q)) &:= \left\{u \in L_2(\mathbb{R}) \mid u, u' - Qu \in W^1_{1,loc}(\mathbb{R}),\, l_Q[u] \in L_2(\mathbb{R})\right\}, \end{aligned}$$

and

$$\begin{aligned} \dot{S}_{min}(q)u &:= l_Q[u], \\ \mathrm{Dom}(\dot{S}_{min}(q)) &:= \left\{u \in \mathrm{Dom}(S_{max}(q)) \mid \operatorname{supp} u \Subset \mathbb{R}\right\}. \end{aligned}$$

It is obvious that operators $S_{max}(q)$ are defined on the maximally possible linear manifolds on which quasi-expressions $l_Q[\cdot]$ are well defined.

Proposition 8. *Let $q(x) \in H^{-1}_{per}(\mathbb{R}, \mathbb{R})$. Then the following statements are fulfilled:*

(I) *Operators $\dot{S}_{min}(q)$ are symmetric and lower semibounded on the Hilbert space $L_2(\mathbb{R})$. In particular, they are closable.*

(II) *Closures $S_{min}(q)$ of operators $\dot{S}_{min}(q)$, $S_{min}(q) := (\dot{S}_{min}(q))^{\sim}$, are symmetric, lower semibounded operators on the Hilbert space $L_2(\mathbb{R})$ with deficiency numbers of a view (m, m) where $0 \le m \le 2$. Operators $S_{max}(q)$ are adjoint to operators $S_{min}(q)$,*

$$S^*_{min}(q) = S_{max}(q).$$

In particular, $S_{max}(q)$ are closed operators on the Hilbert space $L_2(\mathbb{R})$, and

$$S^*_{max}(q) = S_{min}(q).$$

(III) *Domains $\mathrm{Dom}(S_{min}(q))$ of operators $S_{min}(q)$ consist of those and only those functions $u \in \mathrm{Dom}(S_{max}(q))$ which satisfy the conditions:*

$$[u, v]_{+\infty} - [u, v]_{-\infty} = 0 \quad \forall v \in \mathrm{Dom}(S_{max}(q)),$$

where the limits

$$[u, v]_{+\infty} := \lim_{x \to +\infty} [u, v]_x, \quad \text{and} \quad [u, v]_{-\infty} := \lim_{x \to -\infty} [u, v]_x$$

are well defined and exist.

Proposition 8, which describes properties of operators $\dot{S}_{min}(q)$ and $S_{max}(q)$, is proved by using methods of a linear quasi-differential operators theory in Appendix A.2.

In Proposition 10 we define the Friedrichs extensions of minimal operators $S_{min}(q)$. But firstly, for a convenience, we remind some related facts and prove the useful Lemma 9.

Let H be a Hilbert space, and $\dot{A}$ is a densely defined, lower semibounded linear operator on H. Hence, $\dot{A}$ is a closable, symmetric operator. Define by A its closure, $A := (\dot{A})^{\sim}$.

Set

$$\dot{t}[u, v] := (\dot{A}u, v), \qquad \mathrm{Dom}(\dot{t}) := \mathrm{Dom}(\dot{A}).$$

As known [Kt] sesqui-linear form $\dot{t}[u, v]$ is closable, lower semibounded and symmetric on a Hilbert space H one. Let $t[u, v]$ be its closure, $t := (\dot{t})^{\sim}$.

By the operator $\dot{A}$ it is uniquely defined its Friedrichs extension A_F [Kt],

$$t[u, v] = (A_F u, v), \qquad u \in \mathrm{Dom}(A_F) \subset \mathrm{Dom}(t),\, v \in \mathrm{Dom}(t).$$

Due to the First Representation Theorem [Kt] operator A_F is lower semibounded and self-adjoint. In Lemma 9 we describe its domain, but at first note that the following relationships hold:

$$\dot{A} \subset A \subset A_F \subset A^*.$$

Lemma 9. *Let A_F be a Friedrichs extension of a densely defined, lower semibounded operator $\dot{A}$ on a Hilbert space H, and let $t[u,v]$ is a densely defined, closed, symmetric and bounded from below on H sesqui-linear form built by operator $\dot{A}$. Then the following formula*

$$\mathrm{Dom}(A_F) = \mathrm{Dom}(t) \cap \mathrm{Dom}(A^*)$$

holds.

Proof. It is obvious that

$$\mathrm{Dom}(A_F) \subset \mathrm{Dom}(t) \cap \mathrm{Dom}(A^*).$$

Let prove the inverse inclusion.

Let $u \in \mathrm{Dom}(t) \cap \mathrm{Dom}(A^*)$, and $v \in \mathrm{Dom}(\dot{A}) \subset \mathrm{Dom}(A_F) \subset \mathrm{Dom}(t)$. Remark that $\mathrm{Dom}(\dot{A})$ is a core of the form $t[u,v]$ as well as $\mathrm{Dom}(t) \cap \mathrm{Dom}(A^*)$ containing $\mathrm{Dom}(\dot{A})$. Then we have

$$(A^*u, v) = (u, \dot{A}v) = (u, A_F v) = \overline{(A_F v, u)} = \overline{t[v,u]} = t[u,v],$$

i.e.,

$$t[u,v] = (A^*u, v), \qquad u \in \mathrm{Dom}(t) \cap \mathrm{Dom}(A^*),\ v \in \mathrm{Dom}(\dot{A}).$$

Due to the First Representation Theorem [Kt] we get that $u \in \mathrm{Dom}(A_F)$, i.e.,

$$\mathrm{Dom}(t) \cap \mathrm{Dom}(A^*) \subset \mathrm{Dom}(A_F).$$

The proof is complete. □

Proposition 10. *The Friedrichs extensions $S_F(q)$ of the operators $S_{min}(q)$ are defined in the following fashion:*

$$S_F(q)u := l_Q[u],$$
$$\mathrm{Dom}(S_F(q)) := \left\{ u \in H^1(\mathbb{R}) \mid u' - Qu \in W^1_{1,loc}(\mathbb{R}),\ l_Q[u] \in L_2(\mathbb{R}) \right\}.$$

Proof. Let us introduce the sesqui-linear forms,

$$\dot{t}[u,v] := (\dot{S}_{min}(q)u, v), \quad \mathrm{Dom}(\dot{t}) := \mathrm{Dom}(\dot{S}_{min}(q)).$$

As well known [Kt] the sesqui-linear forms $\dot{t}[u,v]$ are densely defined, closable, symmetric and bounded from below on the Hilbert space $L_2(\mathbb{R})$. Taking into account that $\mathrm{Dom}(\dot{S}_{min}(q)) \subset H^1_{comp}(\mathbb{R})$ the forms $\dot{t}[u,v]$ have got a view

$$\dot{t}[u,v] = (u', v') - (Qu, v') - (Qu', v), \quad \mathrm{Dom}(\dot{t}) \subset H^1_{comp}(\mathbb{R}).$$

Set

$$\dot{t}_1[u,v] := (u', v') + (u, v), \qquad \mathrm{Dom}(\dot{t}_1) := \mathrm{Dom}(\dot{S}_{min}(q)) \subset H^1_{comp}(\mathbb{R}),$$
$$\dot{t}_2[u,v] := -(Qu, v') - (Qu', v) - (u, v), \qquad \mathrm{Dom}(\dot{t}_2) := \mathrm{Dom}(\dot{S}_{min}(q)) \subset H^1_{comp}(\mathbb{R}),$$

i.e.,

$$\dot{t} = \dot{t}_1 + \dot{t}_2.$$

As well known the form $\dot{t}_1[u,v]$ is closable, and its closure $t_1[u,v]$, $t_1 := (\dot{t}_1)^{\sim}$, has the representation

$$t_1[u,v] = (u', v') + (u, v), \quad \mathrm{Dom}(t_1) = H^1(\mathbb{R}).$$

As it was shown in the [HrMk] the forms $\dot{t}_2[u,v]$ are t_1-bounded with relative boundary 0. So, we finally obtain that the forms $\dot{t}[u,v]$ closures $t[u,v]$, $t := (\dot{t})^{\sim}$, are defined as following,

$$t[u,v] = (u', v') - (Qu, v') - (Qu', v), \quad \mathrm{Dom}(t) = H^1(\mathbb{R}).$$

And the sesqui-linear forms $t[u,v]$ are densely defined, closed, symmetric and lower semi-bounded on the Hilbert space $L_2(\mathbb{R})$.

Further, as

$$\begin{aligned} S^*_{min}(q)u &= l_Q[u], \\ \mathrm{Dom}(S^*_{min}(q)) &= \left\{u \in L_2(\mathbb{R}) \mid u, u' - Qu \in W^1_{1,loc}(\mathbb{R}),\, l_Q[u] \in L_2(\mathbb{R})\right\}, \end{aligned}$$

applying Lemma 9 we get the need representations for Friedrichs extensions of operators $\dot{S}_{min}(q)$.

The proof is complete. □

Statement 11. *It is valid the following envelops:*

$$\dot{S}_{min}(q) \subset S_{min}(q) \subset S_F(q) \subset S_{max}(q)$$

and

$$\begin{gathered} \mathrm{Dom}(\dot{S}_{min}(q)) \subset H^1_{comp}(\mathbb{R}), \\ \mathrm{Dom}(S_{min}(q)) \subset H^1(\mathbb{R}), \quad \mathrm{Dom}(S_F(q)) \subset H^1(\mathbb{R}), \\ \mathrm{Dom}(S_{max}(q)) \subset L_2(\mathbb{R}) \cap H^1_{loc}(\mathbb{R}). \end{gathered}$$

Statement 11 immediately follows from the correspondent definitions and not very complicate computations.

Now, our aim is to prove a self-adjointness of maximal quasi-differential operators $S_{max}(q)$.

Proposition 12. *Let $q(x) \in H^{-1}_{per}(\mathbb{R},\mathbb{R})$. The following statements are equivalent:*

(a) *Operators $S_{max}(q)$ are self-adjoint.*
(b) $\mathrm{Dom}(S_{max}(q)) \subset H^1(\mathbb{R})$.
(c) $u' - Qu \in L_2(\mathbb{R}) \cap W^1_{1,loc}(\mathbb{R}) \quad \forall u \in \mathrm{Dom}(S_{max}(q))$.

Proof. (a) Let $S_{max}(q)$ are self-adjoint. Then as it follows from Proposition 8.II and Statement 11 we obtain

$$\begin{aligned} &S_{min}(q) = S_F(q) = S_{max}(q), \\ &\mathrm{Dom}(S_{min}(q)) = \mathrm{Dom}(S_F(q)) = \mathrm{Dom}(S_{max}(q)) \subset H^1(\mathbb{R}), \end{aligned}$$

and (b) is true.

Further, under the assumptions $Q \in L_{2,per}(\mathbb{R})$ and $u \in H^1(\mathbb{R})$ we have got $Qu \in L_2(\mathbb{R})$ [HrMk] that yields (c).

(b) Let suppose that $\mathrm{Dom}(S_{max}(q)) \subset H^1(\mathbb{R})$. As above we get $Qu \in L_2(\mathbb{R})$, and as a consequence we obtain (c). Then the statement (a) follows from the Lagrange Identity (4), taking into account that

$$[u,v]_{+\infty} = 0 \quad \text{and} \quad [u,v]_{-\infty} = 0$$

for $u, v \in L_2(\mathbb{R})$ and $u' - Qu, v' - Qv \in L_2(\mathbb{R}) \cap W^1_{1,loc}(\mathbb{R})$.

(c) Let given (c), i.e., $u' - Qu \in L_2(\mathbb{R}) \cap W^1_{1,loc}(\mathbb{R}) \quad \forall u \in \mathrm{Dom}(S_{max}(q))$. Then applying the Lagrange Identity (4) as above we have got (a), and as a consequence (b).

The proof is complete. □

Hryniv and Mykytyuk [HrMk] studied operators associated due to the First Representation Theorem [Kt] with the sesqui-linear forms

$$t[u,v] = (u',v') - (Qu,v') - (Qu',v), \quad \mathrm{Dom}(t) = H^1(\mathbb{R}).$$

I.e., they studied namely Friedrichs extensions $S_F(q)$.

Djakov and Mityagin [DjMt] also treated Friedrichs extensions $S_F(q)$ *a priori* considering operators on the domains

$$\operatorname{Dom}(S_F(q)) = \left\{ u \in H^1(\mathbb{R}) \mid u' - Qu \in W^1_{1,loc}(\mathbb{R}),\, l_Q[u] \in L_2(\mathbb{R}) \right\},$$

see Proposition 10 and Proposition 12.

So, due to Proposition 8.II we have

$$S_{max}(q) \supset S^*_{max}(q),$$

and therefore it remains to show a symmetry of the operators $S_{max}(q)$,

$$S_{max}(q) \subset S^*_{max}(q).$$

We do it applying Lemma 5.

Let consider on the Hilbert space $L_2(\mathbb{R})$ a shift operator

$$(Uf)(x) := f(x+1), \qquad \operatorname{Dom}(U) := L_2(\mathbb{R}),$$

then $\sigma_p(U) = \varnothing$.

Further, let $f \in \operatorname{Dom}(S_{max}(q))$. It is obvious that $Uf \in \operatorname{Dom}(S_{max}(q))$ also, and

$$U(S_{max}(q)f) = Ul_Q[f(x)] = l_Q[f(x+1)] = l_Q[(Uf)(x)] = S_{max}(q)(Uf),$$

i.e., operators $S_{max}(q)$ and U commute.

Taking into account that $S_{max}(q)$ are the second order quasi-differential operators, i.e., their possible eigenvalues cannot have multiplicities more than two, and applying Lemma 5 to the operators $S_{max}(q)$ and U we obtain the next proposition.

Proposition 13. *Point spectra $\sigma_p(S_{max}(q))$ of the quasi-differential operators $S_{max}(q)$ are empty.*

Theorem 14. *The quasi-differential operators $S_{max}(q)$ are self-adjoint.*

Proof. From Proposition 8.II and Proposition 13 we get that the minimal symmetric operators $S_{min}(q)$ have deficiency index of a view $(0,0)$, i.e., they are self-adjoint. Due to Proposition 8.II this implies that the operators $S_{max}(q)$ are self-adjoint also.

The proof is complete. □

Corollary 15. *Minimal operators $S_{min}(q)$, Friedrichs extensions $S_F(q)$ and maximal operators $S_{max}(q)$ coincide. In particular, they are self-adjoint and lower semibounded.*

Corollary 16. *Let $q(x) \in H^{-1}_{per}(\mathbb{R},\mathbb{R})$, and $q_n(x) \in H^{-1}_{per}(\mathbb{R},\mathbb{R})$, $n \in \mathbb{N}$ such that*

$$q_n(x) \overset{H^{-1}_{per}(\mathbb{R})}{\longrightarrow} q(x) \quad as \quad n \to \infty.$$

Then the Hill-Schrödinger operators $S(q_n)$, $n \in \mathbb{N}$ converge to the operators $S(q)$ in the norm resolvent sense,

$$\left\| (S(q_n) - \lambda I)^{-1} - (S(q) - \lambda I)^{-1} \right\| \to 0 \quad as \quad n \to \infty,$$

for any λ belonging to resolvent sets of $S(q)$ and $S(q_n)$, $n \in \mathbb{N}$.

Proof. It immediately follows from [HrMk, Theorem 4.1] and Corollary 15. □

In particular, the Hill-Schrödinger operators $S(q)$ with distributional potentials $q(x) \in H^{-1}_{per}(\mathbb{R},\mathbb{R})$ are a sequence limits $S_{lim}(q)$ of operators $S(q_n)$, $n \in \mathbb{N}$ with smooth potentials $q_n(x) \in L_{2,per}(\mathbb{R},\mathbb{R})$. For an instance, for

$$q(x) = \sum_{k \in \mathbb{Z}} \widehat{q}(2k)\, e^{i\,2k\pi x} \in H^{-1}_{per}(\mathbb{R},\mathbb{R})$$

we may choose

$$q_n(x) := \sum_{|k|\le n} \widehat{q}(2k)\, e^{i\,2k\pi x} \in C^{\infty}_{per}\left(\mathbb{R},\mathbb{R}\right), \quad n\in\mathbb{N}.$$

Now, we are going to define the Hill-Schrödinger operators with distributional potentials as form-sum operators [Krt]. We will show that this definition coincides with the given above ones.

Let consider on the Hilbert space $L_2(\mathbb{R})$ the sesqui-linear forms

$$\tau[u,v] := \left\langle -\frac{d^2}{dx^2}u, v\right\rangle_{L_2(\mathbb{R})} + \langle q(x)u, v\rangle_{L_2(\mathbb{R})}, \quad \mathrm{Dom}(\tau) = H^1(\mathbb{R}),$$

generated by the one-dimensional Schrödinger operators with $q(x)\in H^{-1}_{per}(\mathbb{R},\mathbb{R})$.

Here, by $\langle\cdot,\cdot\rangle_{L_2(\mathbb{R})}$ is denoted sesqui-linear form pairing dual, respectively $L_2(\mathbb{R})$, spaces $H^s(\mathbb{R})$ and $H^{-s}(\mathbb{R})$ for $s\in\mathbb{R}$, which (sesqui-linear form) is an extension by continuity the $L_2(\mathbb{R})$-inner product [Brz, GrGr],

$$\langle f, g\rangle_{L_2(\mathbb{R})} := \int_{\mathbb{R}} f(x)\overline{g(x)}\,dx \quad \forall f,g\in L_2(\mathbb{R}).$$

As known [Krt] the sesqui-linear forms $\tau[u,v]$ are densely defined, closed, bounded from below, defined on the Hilbert space $L_2(\mathbb{R})$ ones. Due to the First Representation Theorem [Kt] with them it is associated uniquely defined on the Hilbert space $L_2(\mathbb{R})$ self-adjoint, lower semibounded operators $S_{form}(q)$ such that

i) $\mathrm{Dom}\,(S_{form}(q))\subset\mathrm{Dom}\,(\tau)$, and

$$\tau[u,v] = (S_{form}(q)u, v) \quad \forall u\in\mathrm{Dom}\,(S_{form}(q))\,, \forall v\in\mathrm{Dom}\,(\tau)\,;$$

ii) $\mathrm{Dom}\,(S_{form}(q))$ are cores of the forms $\tau[u,v]$;

iii) if $u\in\mathrm{Dom}\,(\tau)$, $w\in L_2(\mathbb{R})$, and

$$\tau[u,v] = (w,v)$$

hold for every v in cores of the forms $\tau[u,v]$, then $u\in\mathrm{Dom}\,(S_{form}(q))$ and

$$S_{form}(q)u = w.$$

Operators $S_{form}(q)$ are called form-sum operators associated with the forms $\tau[u,v]$, and denoted as following

$$S_{form}(q) := -\frac{d^2}{dx^2} \dotplus q(x),$$

and also it is convenient to use the denotations

$$\tau_{S_{form}(q)}[u,v] \equiv \tau[u,v].$$

Proposition 17 ([Krt]). *The Hill-Schrödinger operators with distributional potentials from the negative Sobolev space $H^{-1}_{per}(\mathbb{R},\mathbb{R})$ are well defined on the Hilbert space $L_2(\mathbb{R})$ as self-adjoint, lower semibounded form-sum operators $S_{form}(q)$,*

$$S_{form}(q) = -\frac{d^2}{dx^2} \dotplus q(x),$$

associated with the sesqui-linear forms

$$\tau_{S_{form}(q)}[u,v] = \left\langle -\frac{d^2}{dx^2}u, v\right\rangle_{L_2(\mathbb{R})} + \langle q(x)u, v\rangle_{L_2(\mathbb{R})}, \quad \mathrm{Dom}(\tau) = H^1(\mathbb{R}),$$

acting on the dense domains

$$\mathrm{Dom}\,(S_{form}(q)) := \left\{ u\in H^1(\mathbb{R}) \,\middle|\, -\frac{d^2}{dx^2}u + q(x)u \in L_2(\mathbb{R}) \right\}$$

as

$$S_{form}(q)u := -\frac{d^2}{dx^2}u + q(x)u \in L_2(\mathbb{R}), \quad u \in \operatorname{Dom}(S_{form}(q)).$$

Theorem 18. *The quasi-differential operators $S(q)$ and form-sum operators $S_{form}(q)$ coincide.*

Proof. Let $u \in \operatorname{Dom}(S(q))$. Let us remember that

$$\operatorname{Dom}(S(q)) = \left\{u \in H^1(\mathbb{R}) \mid u' - Qu \in W^1_{1,loc}(\mathbb{R}),\, l_Q[u] \in L_2(\mathbb{R})\right\},$$

i.e.,

$$\operatorname{Dom}(S(q)) \subset \operatorname{Dom}(\tau_{S_{form}(q)}) = H^1(\mathbb{R}).$$

Then we have,

$$\begin{aligned}\tau_{S_{form}(q)}[u,v] &= \langle -u'', v\rangle_{L_2(\mathbb{R})} + \langle q(x)u, v\rangle_{L_2(\mathbb{R})} = \langle u', v'\rangle_{L_2(\mathbb{R})} - \langle Q(x), \overline{u'}v + \overline{u}v'\rangle_{L_2(\mathbb{R})} \\ &= (u', v') - (Qu, v') - (Qu', v) = (l_Q[u], v) \quad \forall v \in C^\infty_{comp}(\mathbb{R}).\end{aligned}$$

And due to the First Representation Theorem [Kt] we conclude that

$$u \in \operatorname{Dom}(S_{form}(q)), \quad \text{and} \quad S_{form}(q)u = l_Q[u],$$

i.e.,

$$S(q) \subset S_{form}(q).$$

Taking into account that operators $S(q)$ and $S_{form}(q)$ are self-adjoint from the latter we have got also the inverse inclusions

$$S(q) \supset S_{form}(q).$$

The proof is complete. □

3.3. **Spectra of the Hill-Schrödinger operators with distributional potentials.** Here, we establish characteristic properties of a spectrum structure of the Hill-Schrödinger operators $S(q)$ with distributional potentials $q(x) \in H^{-1}_{per}(\mathbb{R},\mathbb{R})$. By means of a limit process in generalized sense applied to the Hill-Schrödinger operators $S(q_n)$, $n \in \mathbb{N}$ with smooth potentials $q_n(x) \in L_{2,per}(\mathbb{R},\mathbb{R})$ (see Corollary 16) we show that the Hill-Schrödinger operators $S(q)$ with distributions $q(x) \in H^{-1}_{per}(\mathbb{R},\mathbb{R})$ as potentials have continuous spectra with a band and gap structure.

For different approaches see [HrMk, Krt, DjMt].

At first let us remind well known results related with the classical case of $L_{2,per}(\mathbb{R},\mathbb{R})$-potentials $q(x)$,

$$q(x) \in L_{2,per}(\mathbb{R},\mathbb{R}), \tag{5}$$

see, for an example, [DnSch2, ReSi4]. Under the assumption (5) the Hill-Schrödinger operators $S(q)$ are lower semibounded and self-adjoint on the Hilbert space $L_2(\mathbb{R})$, they have absolutely continuous spectra with a band and gap structure.

Spectra of the Hill-Schrödinger operators are defined well by a location of the spectral gap endpoints. It is known that for the endpoints $\{\lambda_0(q), \lambda^{\pm}_k(q)\}_{k=1}^\infty$ of spectral gaps it is valid the following inequalities:

$$-\infty < \lambda_0(q) < \lambda^-_1(q) \le \lambda^+_1(q) < \lambda^-_2(q) \le \lambda^+_2(q) < \dots. \tag{6}$$

Spectral bands (or stability zones),

$$\mathcal{B}_0(q) := [\lambda_0(q), \lambda^-_1(q)], \qquad \mathcal{B}_k(q) := [\lambda^+_k(q), \lambda^-_{k+1}(q)], \quad k \in \mathbb{N},$$

are characterized as a locus of those real $\lambda \in \mathbb{R}$ for which all solutions of the equation

$$S(q)u = \lambda u \tag{7}$$

are bounded. On the other hand, spectral gaps (or instability zones),

$$\mathcal{G}_0(q) := (-\infty, \lambda_0(q)), \qquad \mathcal{G}_k(q) := (\lambda_k^-(q), \lambda_k^+(q)), \quad k \in \mathbb{N},$$

are a locus of those real $\lambda \in \mathbb{R}$ for which any nontrivial solution of the equation (7) is unbounded.

As we see from the (6) it is possible situation when

$$\lambda_k^-(q) = \lambda_k^+(q)$$

for some $k \in \mathbb{N}$. In this case one say that the correspondent spectral gap $\mathcal{G}_k(q)$ is *collapsed*, or *closed*. Note, that for spectral bands it cannot happen.

Further, it happens that the endpoints of spectral gaps for even numbers $k \in \mathbb{Z}_+$ are periodic eigenvalues of the problems on the interval $[0,1]$:

$$S_+(q)u := -u'' + q(x)u = \lambda u, \quad u \in \operatorname{Dom}(S_+(q)),$$

and the endpoints of spectral gaps for odd numbers $k \in \mathbb{N}$ are semiperiodic eigenvalues of the problems on the interval $[0,1]$:

$$S_-(q)u := -u'' + q(x)u = \lambda u, \quad u \in \operatorname{Dom}(S_-(q)).$$

Under the assumption (5) domains of operators $S_+(q)$ and $S_-(q)$ have a view

$$\operatorname{Dom}(S_\pm(q)) = \left\{ u \in H^2[0,1] \,\middle|\, u^{(j)}(0) = \pm u^{(j)}(1),\ j = 0,1 \right\}.$$

Now, applying a limit process in generalized sense (see Corollary 16) to the Hill-Schrödinger operators $S(q_n)$, $n \in \mathbb{N}$ with $L_{2,per}(\mathbb{R},\mathbb{R})$-potentials $q_n(x)$ we establish the following statement.

Theorem 19. *Suppose that $q(x) \in H_{per}^{-1}(\mathbb{R},\mathbb{R})$. Then the Hill-Schrödinger operators $S(q)$ have continuous spectra with a band and gap structure such that the endpoints $\{\lambda_0(q), \lambda_k^\pm(q)\}_{k=1}^\infty$ of spectral gaps satisfy the inequalities:*

$$-\infty < \lambda_0(q) < \lambda_1^-(q) \le \lambda_1^+(q) < \lambda_2^-(q) \le \lambda_2^+(q) < \dots .$$

Moreover, the endpoints of spectral gaps for even/odd numbers $k \in \mathbb{Z}_+$ are periodic/semiperiodic eigenvalues of the problems on the interval $[0,1]$,

$$S_\pm(q)u = -u'' + q(x)u = \lambda u, \quad u \in \operatorname{Dom}(S_\pm(q)).$$

Remark 20. Operators $S_+(q)$ and $S_-(q)$ are well defined on the Hilbert space $L_2(0,1)$ as lower semi-bounded, self-adjoint form-sum operators,

$$S_\pm(q) = \left(-\frac{d^2}{dx^2}\right)_\pm \dotplus q(x).$$

Also they can be well defined in alternative equivalent ways: as quasi-differential ones and as a sequence limits in the norm resolvent sense of operators with smooth potentials.

In the papers [MiMl3, MiMl4, MiMl5] the authors meticulously treated the form-sum operators

$$S_\pm(V) = \left((-1)^m \frac{d^{2m}}{dx^{2m}}\right)_\pm \dotplus V(x), \quad V(x) \in H_{per}^{-m}[0,1],\ m \in \mathbb{N}$$

defined on $L_2(0,1)$.

And in the [Mlb, MiMl1, MiMl2] it is studied two terms differential operators of an even order defined on the *negative* Sobolev spaces.

Proof. Let $\{q_n(x)\}_{n\in\mathbb{N}}$ is a sequence of real-valued trigonometric polynomials which converge to the singular potential $q(x)$ by the norm of the space $H^{-1}_{per}(\mathbb{R})$. With this sequence one may associate a sequence of self-adjoint operators $\{S_{\pm}(q_n)\}_{n\in\mathbb{N}}$ defined on $L_2(0,1)$ and a sequence of Hill operators $\{S(q_n)\}_{n\in\mathbb{N}}$ defined on $L_2(\mathbb{R})$. As it was proved by the authors in the [MiMl3, MiMl5] the sequences $\{S_{\pm}(q_n)\}_{n\in\mathbb{N}}$ converge to operators $S_{\pm}(q)$ in the norm resolvent sense. Hence, eigenvalues of these operators $\{S_{\pm}(q_n)\}_{n\in\mathbb{N}}$ converge to correspondent eigenvalues of limiting operators $S_{\pm}(q)$ [ReSi1, Theorem VIII.23 and Theorem VIII.24] (also see [Kt]). Further, as well known [CdLv, DnSch2] for the operators $\{S_{\pm}(q_n)\}_{n\in\mathbb{N}}$ it holds the assertions of theorem, i.e.,

$$-\infty < \lambda_0(q_n) < \lambda_1^-(q_n) \le \lambda_1^+(q_n) < \lambda_2^-(q_n) \le \lambda_2^+(q_n) < \dots . \tag{8}$$

Moreover, as we already have proved (see Corollary 16) the sequence $\{S(q_n)\}_{n\in\mathbb{N}}$ converge to the operator $S(q)$ in the norm resolvent sense. Therefore, from the (8) we get

$$-\infty < \lambda_0(q) \le \lambda_1^-(q) \le \lambda_1^+(q) \le \lambda_2^-(q) \le \lambda_2^+(q) \le \dots ,$$

where $\lambda_0(q)$, $\lambda_{2k}^{\pm}(q) \in \sigma(S_+(q))$ and $\lambda_{2k-1}^{\pm}(q) \in \sigma(S_-(q))$, $k \in \mathbb{N}$.

Now, it remains to show that the strong inequalities

$$\lambda_k^+(q_n) < \lambda_{k+1}^-(q_n), \quad k \in \mathbb{Z}_+$$

can not become into equalities. Really, let suppose contrary. Then, one of the operator $S(q)$ spectral zones degenerates into the point:

$$\lambda_{k_0}^+(q) = \lambda_{k_0+1}^-(q), \quad k_0 \in \mathbb{Z}_+.$$

As it is isolate the operator $S(q)$ spectrum point, therefore, it cannot belong to continuous spectrum $\sigma_c(S(q))$ of the one. On the other hand, it cannot belong to the operator $S(q)$ point spectrum as $\sigma_p(S(q)) = \varnothing$. And obtained contradiction proves the inequalities of theorem.

The proof is complete. □

4. Concluding remarks

From the direct integral decomposition of the Hill-Schrödinger operators $S(q)$ [HrMk] and the known Reed-Simon Theorem [ReSi4, Theorem XIII.86] follow that $\sigma_{sc}(S(q)) = \varnothing$. Therefore, the proved in this paper continuity of the operators $S(q)$ spectra yield their absolutely continuity [MiSb].

From Theorem C and the authors results [MiMl3] one obtain a series of theorems establishing relationships between spectral gap lengths and a smoothness of distributional potentials $q(x) \in H^{-s}_{per}(\mathbb{R},\mathbb{R})$, $s \ge -1$ of the Hill-Schrödinger operators $S(q)$ [MiMl6].

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Appendix: Some proofs

A.1. **Proof of Statement 4.** At first note that from the relationships

$$\dot{L}_{min}(q) \subset L_{min}(q)$$

follows that

$$(\dot{L}_{min}(q))^{\sim} \subset L_{min}(q),$$

see Proposition 3.III. Therefore, it suffices to show the inverse inclusions

$$(\dot{L}_{min}(q))^{\sim} \supset L_{min}(q).$$

Let $\Delta=[\alpha,\beta]$ denotes a fixed, closed interval completely belonging to the interval $[0,1]$, and let

$$\mathfrak{H}_\Delta := L_2(\alpha,\beta).$$

On the Hilbert space $\mathfrak{H}_\Delta$ one may consider operators $L_{min,\Delta}(q)$ and $L_{max,\Delta}(q)$ generated by $l_Q[\cdot]$ on the interval Δ which due to Proposition 3.III are mutually adjoint,

$$L^*_{min,\Delta}(q)=L_{max,\Delta}(q), \qquad L^*_{max,\Delta}(q)=L_{min,\Delta}(q).$$

On the other hand the Hilbert space $\mathfrak{H}_\Delta$ can be well inject into the space $\mathfrak{H}:=L_2(0,1)$ assuming that over the interval Δ a function $u\in\mathfrak{H}_\Delta$ is equal zero. Thus, domains $\mathrm{Dom}(L_{min,\Delta}(q))$ of operators $L_{min,\Delta}(q)$ become of a part of domains $\mathrm{Dom}(L_{max}(q))$ of operators $L_{max}(q)$ as under such extension of function $u\in\mathrm{Dom}(L_{min,\Delta}(q))$ over the interval Δ a continuity of its quasi-derivatives $u^{[j]}(x)$, $j=0,1$, is not destroy. Moreover, extended in such fashion function $u\in\mathrm{Dom}(L_{min,\Delta}(q))$ then belong to the $\mathrm{Dom}(\dot{L}_{min}(q))$. Therefore, if $v\in\mathrm{Dom}(\dot{L}^*_{min}(q))$ then we have

$$\left(\dot{L}^*_{min}(q)v,u\right)=\left(v,\dot{L}_{min}(q)u\right) \qquad \forall u\in\mathrm{Dom}(L_{min,\Delta}(q)). \tag{9}$$

As $u(x)=0$ over the interval Δ scalar product in the (9) is an $\mathfrak{H}_\Delta$-inner product. Denoting these scalar products by index Δ we can re-write (9) as following,

$$\left((\dot{L}^*_{min}(q)v)_\Delta,u\right)_\Delta=(v_\Delta,L_{min,\Delta}(q)u)_\Delta \qquad \forall u\in\mathrm{Dom}(L_{min,\Delta}(q)).$$

Here, by $(\dot{L}^*_{min}(q)v)_\Delta$, v_Δ are denoted functions $\dot{L}^*_{min}(q)v$ and v considered only in the interval Δ. So, from the latter we obtain

$$v_\Delta\in\mathrm{Dom}(L^*_{min,\Delta}(q))=\mathrm{Dom}(L_{max,\Delta}(q))$$

and

$$(\dot{L}^*_{min}(q)v)_\Delta=L^*_{min,\Delta}(q)v_\Delta=L_{max,\Delta}(q)v_\Delta=\left(l_Q[v]\right)_\Delta.$$

As these relationships are valid for any interval $\Delta\subset[0,1]$ we conclude that

$$v\in\mathrm{Dom}(L_{max}(q)), \quad \text{and} \quad \dot{L}^*_{min}(q)v=l_Q[v]=L_{max}(q)v.$$

Thus, it has been proved

$$\dot{L}^*_{min}(q)\subset L_{max}(q),$$

i.e., we have

$$\dot{L}^{**}_{min}(q)\supset L^*_{max}(q)=L_{min}(q),$$

that implies the required inclusions,

$$(\dot{L}_{min}(q))^\sim\supset L_{min}(q).$$

The proof is complete.

A.2. **Proof of Proposition 8.** (I) At first note that

$$\mathrm{Dom}(\dot{S}_{min}(q))\subset H^1_{comp}(\mathbb{R}). \tag{10}$$

Let $u\in\mathrm{Dom}(\dot{S}_{min}(q))$, then we have

$$(\dot{S}_{min}(q)u,u)=(l_Q[u],u)=(u',u')-(Qu,u')-(Qu',u)$$

taking into account that due to the (10)

$$|u'|^2,\, Quu'\in L_{1,comp}(\mathbb{R}).$$

Further, (Qu,u') and (Qu',u) we estimate as in the [HrMk],

$$|(Qu,u')|\le\|Q\|_{L_{2,per}(\mathbb{R})}\left(\varepsilon\|u'\|_{L_2(\mathbb{R})}+b(\varepsilon^{-1})\|u\|_{L_2(\mathbb{R})}\right), \quad \varepsilon\in(0,1], b\ge0,$$

that yields

$$(\dot{S}_{min}(q)u,u) \geq -\gamma(\varepsilon^{-1})\|u\|_{L_2(\mathbb{R})} \quad \forall u \in \mathrm{Dom}(\dot{S}_{min}(q)),\, \gamma \geq 0.$$

And we can conclude that $\dot{S}_{min}(q)$ are Hermitian lower semibounded on $L_2(\mathbb{R})$ operators.

Now, let show that $\mathrm{Dom}(\dot{S}_{min}(q))$ are dense in the Hilbert space $L_2(\mathbb{R})$.

Obviously, it is sufficient to prove that any element $h \in \mathfrak{H}$, $\mathfrak{H} := L_2(\mathbb{R})$, which is orthogonal to $\mathrm{Dom}(\dot{S}_{min}(q))$ is equal zero. Let suppose that $h(x)$ is namely a such function, i.e.,

$$h(x) \perp \mathrm{Dom}(\dot{S}_{min}(q)),$$

and let $\Delta = [\alpha,\beta]$ is a fixed, closed interval compactly belonging to the real axis $\mathbb{R}$ ($\Delta \Subset \mathbb{R}$). Any element $u \in \mathrm{Dom}(S_{min,\Delta}(q))$ can be viewed as element from $\mathrm{Dom}(\dot{S}_{min}(q))$ (with respect to the denotations see the proof of Statement 4), consequently, $h(x)$ is orthogonal to $\mathrm{Dom}(S_{min,\Delta}(q))$. Due to Proposition 3.II $\mathrm{Dom}(S_{min,\Delta}(q))$ are dense in $\mathfrak{H}_\Delta = L_2(\alpha,\beta)$, hence function $h(x)$ considered in the interval Δ has to be equal zero almost everywhere in Δ.

For an arbitrarity of the interval $\Delta \Subset \mathbb{R}$ choice we conclude that $h(x) = 0$ almost everywhere on $\mathbb{R}$.

So, statement (I) of Proposition 8 has been proved completely.

(II) It is obvious that operators $S_{min}(q)$ are symmetric, lower semibouded on the Hilbert space $L_2(\mathbb{R})$ ones.

Make sure that operators $S_{min}(q)$ and $S_{max}(q)$ are mutually adjoint. As $(\dot{S}_{min}(q))^{\sim} = S_{min}(q)$, therefore $\dot{S}^*_{min}(q) = S^*_{min}(q)$, and it suffices to show that

$$\dot{S}^*_{min}(q) = S_{max}(q).$$

Applying the Lagrange Identity (4) we have got

$$(S_{max}(q)u,v) = (u,\dot{S}_{min}(q)v) \quad \forall u \in \mathrm{Dom}(S_{max}(q)),\, \forall v \in \mathrm{Dom}(\dot{S}_{min}(q)),$$

that implies

$$S_{max}(q) \subset \dot{S}^*_{min}(q).$$

So, it remains to prove the inverse inclusions,

$$S_{max}(q) \supset \dot{S}^*_{min}(q).$$

We do it in a similar fashion as in the proof of Statement 4.

Let $v(x)$ is an arbitrary element from the domains $\mathrm{Dom}(\dot{S}^*_{min}(q))$ of the operators $\dot{S}^*_{min}(q)$, and let $\Delta = [\alpha,\beta]$ is a fixed, compact interval ($\Delta \Subset \mathbb{R}$). As in the proof of Statement 4 we obtain

$$\left((\dot{S}^*_{min}(q)v)_\Delta, u\right)_\Delta = (v_\Delta, S_{min,\Delta}(q)u)_\Delta \qquad \forall u \in \mathrm{Dom}(S_{min,\Delta}(q)).$$

So, one can conclude that

$$v_\Delta \in \mathrm{Dom}(S_{max,\Delta}(q))$$

and

$$(\dot{S}^*_{min}(q)v)_\Delta = S^*_{min,\Delta}(q)v_\Delta = S_{max,\Delta}(q)v_\Delta = \left(l_Q[v]\right)_\Delta.$$

Taking into account an arbitrarity of the interval $\Delta \subset \mathbb{R}$ choice we finally get that

$$v \in \mathrm{Dom}(S_{max}(q)), \quad \text{and} \quad \dot{S}^*_{min}(q)v = l_Q[v] = S_{max}(q)v,$$

i.e, the required inclusions

$$S_{max}(q) \supset \dot{S}^*_{min}(q)$$

hold.

Further, let find deficiency index of the operators $S_{min}(q)$. At first it is necessary note as the operators $S_{min}(q)$ are lower semibounded therefore their deficiency numbers are equal.

Let $\lambda \in \mathbb{C}$, $\operatorname{Im}\lambda \neq 0$. Then the operators $S_{min}(q)$ deficiency numbers, which we will denote by m, are equal to the number of linear independent solutions of the equation

$$S^*_{min}(q)u = \lambda u,$$

i.e., of the equation (Proposition 8.II)

$$S_{max}(q)u = \lambda u.$$

In other words the deficiency number is a maximal number of linear independent solutions of the equation

$$l_Q[u] = \lambda u$$

in the Hilbert space $L_2(\mathbb{R})$. As a whole number of linear independent solutions of this equation is equal 2 we conclude that

$$0 \leq m \leq 2.$$

Assertion (II) has been proved.

(III) Let $u, v \in \operatorname{Dom}(S_{max}(q))$. Then applying the Lagrange Identity (4) we conclude that there exist the limits

$$[u,v]_{+\infty} := \lim_{x\to+\infty}[u,v]_x, \quad \text{and} \quad [u,v]_{-\infty} := \lim_{x\to-\infty}[u,v]_x,$$

and as a consequence the Lagrange Identity (4) has got a view

$$(l_Q[u], v) - (u, l_Q[v]) = [u,v]_{-\infty}^{+\infty} \quad \forall u, v \in \operatorname{Dom}(S_{max}(q)). \tag{11}$$

Further, due to Proposition 8.II it holds

$$S_{min}(q) = S^*_{max}(q).$$

Therefore, domains $\operatorname{Dom}(S_{min}(q))$ consist of that and only that functions $u \in \operatorname{Dom}(S_{max}(q))$ which satisfy the relationships

$$(u, S_{max}(q)v) = (S_{max}(q)u, v) \quad \forall v \in \operatorname{Dom}(S_{max}(q)).$$

Together with the Lagrange Identity (11) the latter implies the required assertion, i.e.,

$$u \in \operatorname{Dom}(S_{min}(q)) \Leftrightarrow [u,v]_{+\infty} - [u,v]_{-\infty} = 0 \quad u \in \operatorname{Dom}(S_{max}(q)), \forall v \in \operatorname{Dom}(S_{max}(q)).$$

Proposition 8 has been proved.

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